

# Invariance of the restricted $p$ -power map on integrable derivations under stable equivalences

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## Abstract

We show that the  $p$ -power maps in the first Hochschild cohomology space of finite-dimensional selfinjective algebras over a field of prime characteristic  $p$  commute with stable equivalences of Morita type on the subgroup of classes represented by integrable derivations. We show, by giving an example, that the  $p$ -power maps do not necessarily commute with arbitrary transfer maps in the Hochschild cohomology of symmetric algebras.

## 1 Introduction

Let  $k$  be a field of prime characteristic  $p$ . For symmetric  $k$ -algebras, it is shown in [5] that the Gerstenhaber bracket in Hochschild cohomology commutes with the transfer maps introduced in [7]. Zimmermann proved in [10] that the  $p$ -power map on (the positive part of) Hochschild cohomology commutes with derived equivalences. We show in this paper that the  $p$ -power map, restricted to the classes of integrable derivations, commutes with stable equivalences of Morita type between finite-dimensional selfinjective algebras. We also show, by giving an example, that  $p$ -power maps need not commute with arbitrary transfer maps in the Hochschild cohomology of symmetric algebras. To state our main result, we use the following notation: let  $A$  be a finite-dimensional selfinjective  $k$ -algebra. For  $r$  a positive integer, we denote by  $\text{Aut}_r(A[[t]])$  the subgroup of  $k[[t]]$ -algebra automorphism of  $A[[t]]$  which induce the identity on  $A[[t]]/t^r A[[t]]$ . If  $\alpha \in \text{Aut}_r(A[[t]])$ , then there is a unique  $k[[t]]$ -linear map  $\mu$  on  $A[[t]]$  such that  $\alpha(a) = a + t^r \mu(a)$  for all  $a \in A[[t]]$ .

An easy verification (see Proposition 3.5) shows that the map  $\bar{\mu}$  induced by  $\mu$  on the quotient  $A[[t]]/tA[[t]] \cong A$  is a derivation; any such derivation is called  $r$ -integrable. We denote by  $\text{HH}_r^1(A)$  the image in  $\text{HH}^1(A)$  of all  $r$ -integrable derivations. Let  $A, B$  be finite-dimensional selfinjective  $k$ -algebras,  $M$  be an  $A$ - $B$ -bimodule and  $N$  a  $B$ - $A$ -bimodule. Following Broué [2], we say that  $M$  and  $N$  induce a stable equivalence of Morita type between  $A$  and  $B$  if  $M, N$  are finitely generated projective as left and right modules with the property that  $M \otimes_B N \cong A \oplus X$  for some projective  $A$ - $A$ -bimodule  $X$  and  $N \otimes_A M \cong B \oplus Y$  for some projective  $B$ - $B$ -bimodule  $Y$ . If  $A, B$  are symmetric then  $N$  can be replaced by  $M^\vee$ .

**Theorem 1.1.** *Let  $A, B$  be finite-dimensional selfinjective  $k$ -algebras with separable semisimple quotients, and let  $M, N$  be an  $A$ - $B$ -bimodule,  $B$ - $A$ -bimodule,*

respectively, inducing a stable equivalence of Morita type between  $A$  and  $B$ . For any positive integer  $r$ , the  $p$ -power map sends  $\mathrm{HH}_r^1(A)$  to  $\mathrm{HH}_{rp}^1(A)$ , and we have a commutative diagram of maps

$$\begin{array}{ccc} \mathrm{HH}_r^1(A) & \xrightarrow{\cong} & \mathrm{HH}_r^1(B) \\ \downarrow [p] & & \downarrow [p] \\ \mathrm{HH}_{rp}^1(A) & \xrightarrow{\cong} & \mathrm{HH}_{rp}^1(B) \end{array}$$

where the horizontal isomorphisms are induced by the functor  $N \otimes_A - \otimes_A M$ , and where the vertical maps are the  $p$ -power maps.

In Section 2, we recall some basic results. In Section 3 we prove the main results concerning  $r$ -integrable derivation that allow us to prove in Section 4 the Theorem 1.1. In the last section we provide an example of when the  $p$ -power map does not commute with a transfer map between the Hochschild cohomology of two symmetric algebras.

## 2 Background

Let  $A$  be a finite-dimensional algebra over  $k$ . For any integer  $n \geq 0$  and any  $A \otimes_k A^{op}$ -module  $M$  the Hochschild cohomology of degree  $n$  of  $A$  with coefficients in  $M$  is denoted by  $\mathrm{HH}^n(A; M)$  in particular  $\mathrm{HH}^n(A) = \mathrm{HH}^n(A; A)$ . It is well known that  $\mathrm{HH}^0(A) = Z(A)$  and  $\mathrm{HH}^1(A)$  is the space of derivations modulo inner derivations. The direct sum  $\bigoplus_{n \geq 0} \mathrm{HH}^n(A)$  is a Gerstenhaber algebra, in particular  $\mathrm{HH}^1(A)$  is a Lie algebra. In addition, if the characteristic of  $k$  is positive, there is a map  $[p] : \mathrm{HH}^1(A) \rightarrow \mathrm{HH}^1(A)$ , called  $p$ -power map. This map is induced by the map sending a derivation  $f$  to  $f^p$  that is  $f$  composed  $p$ -times with itself. Then  $\mathrm{HH}^1(A)$  endowed with the  $p$ -power becomes a restricted Lie algebra. Let  $A[[t]]$  be the formal power series with coefficients in  $A$ . By [6, 2.1] the canonical map  $A[[t]] \rightarrow A[[t]]/t^r A[[t]]$  induces an isomorphism

$$\mathrm{HH}^n(A[[t]]; A[[t]]/t^r A[[t]]) \cong \mathrm{HH}^n(A[[t]]/t^r A[[t]]). \quad (1)$$

for all  $n \geq 0$  and  $r > 0$ . The following is well known:

**Lemma 2.1.** *Let  $A$  be a finite-dimensional algebra over  $k$  and let  $A[[t]]$  be the formal power series with coefficients in  $A$ . Then the multiplication in  $A[[t]]$  induce a  $k[[t]]$ -algebra isomorphism  $k[[t]] \otimes_k A \cong A[[t]]$ .*

*Proof.* The isomorphism sends  $\sum_{i \geq 0} \lambda_i t^i \otimes a$  to  $\sum_{i \geq 0} \lambda_i a t^i$  where  $\lambda_i \in k$  and  $a \in A$ . In order to show that this is an isomorphism, we construct its inverse as follows: let  $\sum_{i \geq 0} a_i t^i \in A[[t]]$  and let  $\{e_j\}_{1 \leq j \leq n}$  be a  $k$ -basis of  $A$ . Write  $a_i = \sum_{j=1}^n \mu_{ij} e_j$  for every non-negative integer  $i$  where  $\mu_{ij} \in k$ . The inverse map sends  $\sum_{i \geq 0} a_i t^i$  to  $\sum_{j=1}^n \left( \sum_{i \geq 0} \mu_{ij} t^i \otimes e_j \right)$ .  $\square$

**Corollary 2.2.** *Let  $A$  be a finite-dimensional algebra over  $k$  and let  $r$  be a positive integer. Then the canonical map  $Z(A[[t]]) \rightarrow Z(A[[t]]/t^r A[[t]])$  is surjective.*

Let  $n$  be an integer. We recall that if

$$0 \longrightarrow X \xrightarrow{\tau} Y \xrightarrow{\sigma} Z \longrightarrow 0$$

is a short exact sequence of cochain complexes with differentials  $\delta, \epsilon, \zeta$  respectively, then this induces a long exact sequence

$$\dots \longrightarrow H^n(X) \xrightarrow{H^n(\tau)} H^n(Y) \xrightarrow{H^n(\sigma)} H^n(Z) \xrightarrow{d^n} H^{n+1}(X) \longrightarrow \dots$$

depending functorially on the short exact sequence, where  $d^n$  is called the connecting homomorphism which is obtained in the following way: let  $\bar{z} = z + \text{Im}(\zeta^{n-1}) \in H^n(Z)$  for some  $z \in \text{Ker}(\zeta^n) \subseteq Z^n$ . Since  $\sigma$  is surjective in each degree there is  $y \in Y^n$  such that  $\sigma^n(y) = z$ . Then  $\epsilon(y) \in Y^{n+1}$  satisfies

$$\sigma^{n+1}(\epsilon^n(y)) = \zeta^n(\sigma^n(y)) = \zeta^n(z) = 0 \quad (2)$$

Hence  $\epsilon(y) \in \text{ker}(\sigma^{n+1}) = \text{Im}(\tau^{n+1})$ . Thus there is an  $x \in X^{n+1}$  such that  $\tau^{n+1}(x) = \epsilon^n(y)$ . It is easy to check that  $x \in \text{Ker}(\delta^{n+1})$  and the class  $\bar{x} = x + \text{Im}(\delta^n) \in H^{n+1}(X)$  depends only in the class  $\bar{z}$  of  $z$  in  $H^n(Z)$ . The connecting homomorphism sends  $\bar{z}$  to  $\bar{x}$ .

For the next two sections all the tensor products are over  $k$  unless otherwise specified.

### 3 Integrable derivations of degree $r$

**Definition 3.1.** (cf. [8, 1.1]) Let  $A$  be a finite-dimensional  $k$ -algebra. A *higher derivation*  $\underline{D}$  of  $A$  is a sequence  $\underline{D} = (D_i)_{i \geq 0}$  of  $k$ -linear endomorphisms  $D_i : A \rightarrow A$  such that  $D_0 = \text{Id}$  and  $D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b)$  for all  $n \geq 1$  and all  $a, b \in A$ .

For a fixed positive integer  $r$  we denote by  $\text{Aut}_r(A[[t]])$  the group of all  $k[[t]]$ -algebra automorphism of  $A[[t]]$  which induce the identity on  $A[[t]]/t^r A[[t]]$ . Clearly we have an inclusion  $\text{Aut}_r(A[[t]]) \subseteq \text{Aut}_1(A[[t]])$  for every  $r \geq 1$ . Following [8] any higher derivation  $\underline{D} = (D_i)_{i \geq 0}$  of  $A$  determines a unique automorphism  $\alpha \in \text{Aut}_r(A[[t]])$  satisfying  $\alpha(a) = \sum_{i \geq 0} D_i(a)t^i$  for all  $a \in A$  and vice versa. Note that any  $k[[t]]$ -ring endomorphism of  $A[[t]]$  is determined by its restriction to  $A$ . We denote by  $\text{Out}_r(A[[t]])$  the image of the canonical map  $\varphi : \text{Aut}_r(A[[t]]) \rightarrow \text{Out}(A[[t]])$  and by  $\text{Der}(A)$  the set of derivations over  $A$ .

**Lemma 3.2.** Let  $A$  be a finite-dimensional  $k$ -algebra. Let  $r$  be a positive integer. Then  $\text{Out}_r(A[[t]])$  is the kernel of the canonical group homomorphism

$$\psi : \text{Out}(A[[t]]) \rightarrow \text{Out}(A[[t]]/(t^r A[[t]])). \quad (3)$$

*Proof.* Clearly  $\text{Out}_r(A[[t]]) \subseteq \text{Ker}(\psi)$ . Let  $\alpha$  be a representative of an element in the kernel of  $\psi$ . Then  $\psi(\alpha)$  is given by conjugation with an invertible element  $\bar{u} = u + t^r A[[t]]$  in  $A[[t]]/t^r A[[t]]$  where  $u \in A[[t]]$ . If we denote by  $\overline{A[[t]]} = A[[t]]/t^r A[[t]]$ , since  $\bar{u}$  is invertible in  $\overline{A[[t]]}$ , we have  $\overline{A[[t]]} = \overline{A[[t]]}\bar{u}$ . Then we can lift it to  $A[[t]] = A[[t]]u + t^r A[[t]]$ . By Nakayama's Lemma we have  $A[[t]] =$

$A[[t]]u$  hence  $u$  is invertible. Consequently if we replace  $\alpha$  by  $\alpha$  composed with the conjugation given by  $u^{-1}$  then the resulting automorphism is in the same class as  $\alpha$  and it induces the identity on  $A[[t]]/t^r A[[t]]$ .  $\square$

Slightly extending Matsumura [8] we have the following terminology:

**Definition 3.3.** Let  $A$  be a finite-dimensional  $k$ -algebra and let  $r$  be a positive integer. A derivation  $D \in \text{Der}(A)$  is called  $r$ -integrable if there exists a higher derivation  $\underline{D} = (D_i)_{i \geq 0}$  such that  $D_0 = \text{Id}$ ,  $D_i = 0$  for  $1 \leq i \leq r-1$ , and  $D = D_r$ . We denote by  $\text{Der}_r(A)$  the set of  $r$ -integrable derivations of  $A$  and by  $\text{Inn}_r(A)$  the set of  $r$ -integrable which are inner.

It is easy to check that these are abelian groups, using Proposition 3.4.

Note that for  $r = 1$  we have the usual notion of integrable derivation that is integrable derivations are 1-integrable. We recall from [8, 1.5]:

**Proposition 3.4.** *The set of higher derivation is a group with the product defined on (4). In particular if we let  $\underline{D} = (D_i)_{i \geq 0}$  and  $\underline{D}' = (D'_i)_{i \geq 0}$  be two higher derivations, then:*

$$\underline{D} \circ \underline{D}' = \left( \sum_{i=0}^n D_i \circ D'_{n-i} \right)_{n \geq 0} \quad (4)$$

**Proposition 3.5.** *Let  $A$  be a finite-dimensional  $k$ -algebra. Let  $r$  be a positive integer, let  $\alpha \in \text{Aut}_r(A[[t]])$  and let  $\mu : A[[t]] \rightarrow A[[t]]$  be the unique  $k[[t]]$ -linear map such that  $\alpha(a) = a + t^r \mu(a)$  for all  $a \in A[[t]]$ . Then the following hold:*

(a) *The map  $\bar{\mu} : A \cong A[[t]]/tA[[t]] \rightarrow A \cong A[[t]]/tA[[t]]$  induced by  $\mu$  is a derivation. In addition if  $\alpha$  is an inner automorphism, then  $\bar{\mu} = [\bar{d}, -]$  for some  $\bar{d} \in A$ ; that is  $\bar{\mu}$  is a inner derivation.*

(b) *The class of  $\bar{\mu} \in \text{HH}^1(A)$  depends only on the class of  $\alpha \in \text{Out}(A[[t]])$ .*

*Proof.* Let  $a, b \in A[[t]]$ , since  $\alpha$  is an automorphism we have  $\alpha(ab) = ab + t^r \mu(ab)$  is equal to  $\alpha(a)\alpha(b) = ab + t^r \mu(a) + t^r \mu(b) + t^{2r} \mu(a)\mu(b)$  hence we obtain  $\mu(ab) = a\mu(b) + \mu(a)b + t^r \mu(a)\mu(b)$ . Reducing modulo  $t^r$  we have

$$\mu(ab) = a\mu(b) + \mu(a)b \quad (5)$$

hence  $\bar{\mu}$  is a derivation on  $A$ .

Now suppose that  $\alpha$  is an inner automorphism induced by conjugation by an element  $c \in (A[[t]])^\times$  that is  $\alpha(a) = cac^{-1}$ . Since  $\alpha$  induces the identity on  $A[[t]]/t^r A[[t]]$  then taking the projection of  $\alpha$  in  $A[[t]]/t^r A[[t]]$  we have  $\bar{c}\bar{a}\bar{c}^{-1} = \bar{a}$ , that is  $\bar{c}\bar{a} = \bar{a}\bar{c}$  hence  $\bar{c} \in Z(A[[t]]/t^r A[[t]])^\times$ . Since the map  $Z(A[[t]]) \rightarrow Z(A[[t]]/t^r A[[t]])$  is surjective then there is an element  $z \in Z(A)^\times$  such that  $\bar{z} = \bar{c}$  hence such that  $cz^{-1} \in 1 + t^r A[[t]]$ . So if we replace  $c$  by  $cz^{-1}$  we have  $c = 1 + t^r d$  for some  $d \in A[[t]]$ . If we take an  $a \in A[[t]]$  we have  $cac^{-1} = \alpha(a) = a + t^r \mu(a)$  hence  $ca = ac + t^r \mu(a)c$ , that is  $[c, a] = t^r \mu(a)c$ . Now if we replace  $c$  by  $1 + t^r d$  and we divide by  $t^r$  we obtain

$$[d, a] = \mu(a) + t^r \mu(a)d. \quad (6)$$

Consequently  $[\bar{d}, \bar{a}] = \bar{\mu}(\bar{a})$  hence the result.

For the second part we let  $\alpha_1, \alpha_2$  be two representatives in  $\text{Out}(A[[t]])$  with

induced derivations  $\mu_1, \mu_2$ . Since  $\alpha_1 \circ \alpha_2^{-1} \in \text{Inn}(A[[t]])$  then using Proposition 3.4 and first part of the Proposition we have that  $\mu_1 - \mu_2 \in \text{Inn}(A)$ . Hence the result.  $\square$

An equivalent definition of  $r$ -integrable can be deduced from the following: let  $\alpha \in \text{Aut}_r(A[[t]])$  and let  $a = \sum_{i=0}^{\infty} a_i t^i$ . Then  $\alpha(a) = \sum_{i,n \geq 0} D_n(a_i) t^{i+n} = a + t^r \sum_{k \geq r} \sum_{n, i \geq 1, i+n=k} D_n(a_i) t^{r-i-n}$  since  $D_i = 0$  for  $1 \leq i \leq r-1$ . Hence we can write  $\alpha$  as  $\bar{\alpha}(a) = a + t^r \mu(a)$  where  $\mu$  is a linear endomorphism of  $A[[t]]$ . From Proposition 3.5 the map  $\bar{\mu} : A \rightarrow A$  induced by  $\mu$  is a derivation over  $A$ , in fact,  $\bar{\mu}$  is exactly  $D_r$ . Hence a derivation  $D$  on  $A$  is  $r$ -integrable if there is an algebra automorphism of  $A[[t]]$ , say  $\alpha$ , and a  $k[[t]]$ -linear endomorphism  $\mu$  of  $A[[t]]$  such that  $\alpha(a) = a + t^r \mu(a)$  for all  $a \in A[[t]]$  and such that  $D$  is equal to the map  $\bar{\mu}$  induced by  $\mu$  on  $A \cong A[[t]]/tA[[t]]$ .

**Proposition 3.6.** *Let  $A$  be a finite-dimensional  $k$ -algebra and let  $\alpha \in \text{Aut}_1(A[[t]])$ . Let  $(D_i)_{i \geq 0}$  be a higher derivation satisfying  $\alpha(a) = \sum_{i \geq 0} D_i(a) t^i$  for  $a \in A$ . The map that sends  $\alpha$  to  $\sum_{i \geq 0} D_i t^i$  induces a group homomorphism  $\phi : \text{Aut}_1(A[[t]]) \rightarrow (\text{End}_k(A)[[t]])^\times$ .*

*Proof.* Let  $\beta \in \text{Aut}_1(A[[t]])$ . For  $l \geq 0$  let  $E_l \in \text{End}_k(A)$  such that  $\beta(a) = \sum_{l \geq 0} E_l(a) t^l$ . For all  $a \in A$  let  $\{e_j\}_{1 \leq j \leq n}$  be a  $k$ -basis of  $A$ . For every  $i \geq 0$  define  $\mu_{ij} : A \rightarrow k$  such that  $D_i(a) = \sum_{j=1}^n \mu_{ij}(a) e_j$  where  $a \in A$ . On one side we have:

$$\begin{aligned} (\beta \circ \alpha)(a) &= \beta\left(\sum_{i \geq 0} D_i(a) t^i\right) = \sum_{j=1}^n \beta\left(\sum_{i \geq 0} \mu_{ij}(a) t^i e_j\right) \\ &= \sum_{j=1}^n \sum_{i \geq 0} \mu_{ij}(a) t^i \beta(e_j) = \sum_{l \geq 0} \sum_{i \geq 0} \sum_{j=1}^n \mu_{ij}(a) E_l(e_j) t^{i+l} \end{aligned} \quad (7)$$

where the third equation holds since  $\beta$  is an automorphism over  $k[[t]]$ . If we fix a degree  $m \in \mathbb{N}$  we have

$$\begin{aligned} \sum_{l,i} \sum_{j=1}^n \mu_{ij}(a) E_l(e_j) t^{i+l} &= \sum_{l,i} E_l\left(\sum_{j=1}^n \mu_{ij}(a) e_j\right) t^m \\ &= \sum_{l,i} E_l(D_i(a)) t^m \end{aligned} \quad (8)$$

Hence  $\phi(\beta \circ \alpha)$  in degree  $m$  is equal to  $\sum_{i,l \geq 0} E_l \circ D_i t^m$ . This is clearly equal to the coefficient at  $t^m$  of  $\phi(\beta)\phi(\alpha)$ .  $\square$

**Definition 3.7.** Let  $A$  be a finite-dimensional  $k$ -algebra. Let  $r$  be a positive integer then by  $\text{HH}_r^1(A)$  we denote the quotient  $\text{Der}_r(A)/\text{Inn}_r(A)$ .

Clearly  $\text{HH}_r^1(A)$  can be identified with a subgroup of  $\text{HH}^1(A)$ .

**Proposition 3.8.** *Let  $A$  be a finite-dimensional algebra over  $k$ . Let  $r$  be a positive integer and let  $\alpha \in \text{Aut}_r(A[[t]])$ . Let  $\mu$  the unique  $k[[t]]$ -linear map on  $A[[t]]$  such that  $\alpha(a) = a + t^r \mu(a)$  for all  $a \in A[[t]]$ . We denote by  $\bar{\mu}$  the derivation induced on  $A$  by  $\mu$ .*

(a) The derivation  $\bar{\mu}$  is inner if and only if  $\alpha$  induces an inner automorphism in  $A[[t]]/t^{r+1}A[[t]]$ .

(b) We have the following short exact sequence of groups:

$$1 \longrightarrow \text{Out}_{r+1}(A[[t]]) \longrightarrow \text{Out}_r(A[[t]]) \longrightarrow \text{HH}_r^1(A) \longrightarrow 1$$

*Proof.* Let assume that  $\bar{\mu}$  is inner derivation so  $\bar{\mu} = [\bar{d}, -]$  for some  $d \in A[[t]]$ . We can take  $c = 1 + t^r d$  as in the proof of Proposition 3.5. Then from Equation 6 we can choose  $\tau(a) = -\mu(a)d$  so we have  $[d, a] = \mu(a) - t^r \tau(a)$  and since  $c = 1 + t^r d$  then

$$[c, a] = [1 + t^r d, a] = t^r [d, a] \quad (9)$$

So  $[c, a] = t^r [d, a] = t^r \mu(a) - t^{2r} \tau(a)$ . Hence  $t^r \mu(a) = [c, a] + t^{2r} \tau(a)$ . Consequently  $cac^{-1} = a + t^r \mu(a)c^{-1} - t^{2r} \tau(a)c^{-1}$ . Using the fact that  $\alpha(a) = a + t^r \mu(a)$  it follows that  $\alpha(a) - cac^{-1} = t^r \mu(a)(1 - c^{-1}) + t^{2r} \tau(a)c^{-1}$ . Since  $c$  belongs to  $1 + t^r A[[t]]$ , we have  $c^{-1} \in 1 + t^r A[[t]]$  hence  $1 - c^{-1} \in t^r A[[t]]$ . This shows that  $\alpha(a) - cac^{-1} \in t^{2r} A[[t]] \subset t^{r+1} A[[t]]$ . Consequently  $\alpha$  induces an inner automorphism on  $A[[t]]/t^{r+1}A[[t]]$ .

Conversely, suppose that  $\alpha$  acts as an inner automorphism on  $A[[t]]/t^{r+1}A[[t]]$ . Using the same argument as in Lemma 3.2 we may assume that  $\alpha$  acts as identity on  $A[[t]]/t^{r+1}A[[t]]$  hence it induces an inner derivation on  $A[[t]]/t^{r+1}A[[t]]$ . Hence we can assume  $\alpha$  such that  $\alpha \in \text{Aut}_{r+1}(A[[t]])$ . Hence  $\alpha(a) = a + t^{r+1} \mu'(a)$  for some  $\mu'(a) \in A[[t]]$ , which gives the equality  $\mu(a) = t \mu'(a)$ . Consequently we have that  $\mu$  induces the zero map on  $A$ .

For the second part let  $\beta \in \text{Aut}_r(A)$  such that  $\beta(a) = a + t^r \nu(a)$  for all  $a \in A[[t]]$  and for some linear morphism  $\nu$  on  $A[[t]]$ . From Proposition 3.4 and Proposition 3.5 we have that the class determined by  $\beta \circ \alpha$  in  $\text{HH}^1(A)$  is the class determined by  $\bar{\mu} + \bar{\nu}$ .  $\square$

A way to understand the action of the  $p$ -power map on the integrable derivations is by studying it on  $\text{Aut}_1(A[[t]])$  and then using the homomorphism  $\phi : \text{Aut}_1(A[[t]]) \rightarrow (\text{End}_k(A)[[t]])^\times$ .

**Proposition 3.9.** Let  $\underline{D}$  be a higher derivation and let  $l, n$  be positive integers. The term at  $t^l$  in  $\left(\sum_{i \geq 0} D_i t^i\right)^n$  is equal to

$$\sum_{c=1}^l \binom{n}{c} \sum_{\substack{i_1, \dots, i_c \geq 1 \\ i_1 + \dots + i_c = l}} \prod_{j=1}^c D_{i_j} \quad (10)$$

*Proof.* The term at  $t^l$  in  $\left(\sum_{i \geq 0} D_i t^i\right)^n$  is given by

$$\sum_{\substack{i_1, \dots, i_n \geq 0 \\ i_1 + \dots + i_n = l}} \prod_{j=1}^n D_{i_j}. \quad (11)$$

Let  $c$  be a positive integer. Then for each  $c$ -tuple  $(i'_1, i'_2, \dots, i'_c)$  which has non-zero components and such that  $\sum_{j=1}^c i'_j = l$ , there are  $\binom{n}{c}$  different  $n$ -tuples  $(i_1, i_2, \dots, i_n)$  which have the  $c$  non-zero components of the  $c$ -tuple  $(i'_1, i'_2, \dots, i'_c)$

and rest equal to zero. Since  $D_0 = \text{Id}$  then  $\prod_{j=1}^n D_{i_j} = \prod_{j=1}^c D_{i'_j}$ . For a fixed  $c$  the Equation (11) is given by  $\binom{n}{c} \sum_{\substack{i_1, \dots, i_c \geq 1 \\ i_1 + \dots + i_c = l}} \prod_{j=1}^c D_{i_j}$ . If we sum over all  $c$  we have the result.  $\square$

**Corollary 3.10.** *Let  $A$  be a finite-dimensional  $k$ -algebra and let  $\alpha \in \text{Aut}_r(A[[t]])$  for some positive integer  $r$ . Then  $\alpha^p \in \text{Aut}_{rp}(A[[t]])$ . The  $p$ -power map sends  $\text{HH}_r^1(A)$  to  $\text{HH}_{rp}^1(A)$ , and  $\text{Out}_r(A[[t]])$  to  $\text{Out}_{rp}(A[[t]])$  and we have a commutative diagram*

$$\begin{array}{ccc} \text{Out}_r(A[[t]]) & \xrightarrow{(\ )^p} & \text{Out}_{rp}(A[[t]]) \\ \downarrow & & \downarrow \\ \text{HH}_r^1(A) & \xrightarrow{[p]} & \text{HH}_{rp}^1(A) \end{array}$$

where the vertical maps are from Proposition 3.8 (b),  $(\ )^p$  is the  $p$ -fold composition and  $[p]$  is the  $p$ -power map.

*Proof.* Let  $\alpha \in \text{Aut}_r(A[[t]])$  and let  $D_r$  the derivation in  $\text{Der}_r(A)$ . Let  $\underline{D}'$  be the higher derivation associated to  $\alpha^p$ . Using Proposition 3.9, in degree  $l \leq p-1$  we have:

$$\sum_{c=1}^l \binom{p}{c} \sum_{\substack{i_1, \dots, i_c \geq 1 \\ i_1 + \dots + i_c = l}} \prod_{j=1}^c D_{i_j} t^l = 0 \quad (12)$$

since the binomial coefficient give us multiples of  $p$ . For  $l \geq p$

$$\sum_{c=1}^l \binom{p}{c} \sum_{\substack{i_1, \dots, i_c \geq 1 \\ i_1 + \dots + i_c = l}} \prod_{j=1}^c D_{i_j} t^l = \sum_{\substack{i_1, \dots, i_p \geq 1 \\ i_1 + \dots + i_p = l}} \prod_{j=1}^p D_{i_j} t^l \quad (13)$$

Now we know that each  $D_i$  is zero for  $i = 1, \dots, r-1$  so in order to have an element different from zero we should impose that each  $i_j$  be at least  $r$ . Therefore the sum  $i_1 + \dots + i_p = rp$  that is  $l = rp$  hence the first non-zero coefficient is  $D_r^p$ . Consequently the diagram commutes.  $\square$

## 4 A cohomological interpretation of $r$ -integrable derivations

Integrable derivation can also being interpreted using a cohomological point of view. Starting from the short exact sequence of  $A[[t]]$ - $A[[t]]$ -bimodules:

$$0 \longrightarrow A[[t]] \xrightarrow{t^r} A[[t]] \longrightarrow A[[t]]/t^r A[[t]] \longrightarrow 0$$

after dividing by  $tA[[t]]$  and twisting on the right by the automorphism  $\alpha \in \text{Aut}_r(A[[t]])$  we obtain the short exact sequence:

$$0 \longrightarrow A[[t]]/tA[[t]] \xrightarrow{t^r} (A[[t]]/t^{r+1}A[[t]])_\alpha \rightarrow A[[t]]/t^r A[[t]] \longrightarrow 0$$

since  $\alpha$  induces the identity on  $A[[t]]/t^r A[[t]]$  hence also on  $A[[t]]/tA[[t]]$ .

The following proposition is an adaptation of [6, 4.1] to the situation under consideration.

**Proposition 4.1.** *Let  $A$  be a finite-dimensional algebra over  $k$ . Set  $\hat{A} = A[[t]]$  and set  $\hat{A}^e = \hat{A} \otimes_{k[[t]]} \hat{A}^{op}$ . Let  $\alpha \in \text{Aut}_r(\hat{A})$ . Let  $r$  a positive integer and let  $\mu : A \rightarrow A$  be the unique linear map satisfying  $\alpha(a) = a + t^r \mu(a)$ . Let  $P$  be a projective resolution of  $\hat{A}$  as  $\hat{A}^e$ -module. Applying the functor  $\text{Hom}_{\hat{A}}(P, -)$  to the exact sequence of  $\hat{A}^e$ -modules*

$$0 \longrightarrow \hat{A}/t\hat{A} \xrightarrow{t^r} (\hat{A}/t^{r+1}\hat{A})_\alpha \longrightarrow \hat{A}/t^r \hat{A} \longrightarrow 0$$

*yields a short exact sequence of cochain complexes*

$$0 \longrightarrow \text{Hom}_{\hat{A}^e}(P, A) \xrightarrow{t^r} \text{Hom}_{\hat{A}^e}(P, (\hat{A}/t^{r+1}\hat{A})_\alpha) \rightarrow \text{Hom}_{\hat{A}^e}(P, \hat{A}/t^r \hat{A}) \rightarrow 0$$

*The first non trivial connecting homomorphism can be identified with a map*

$$\text{End}_{\hat{A}^e}(\hat{A}/t^r \hat{A}) \rightarrow \text{HH}^1(A) \quad (14)$$

*and this map sends  $\text{Id}_{\hat{A}/t^r \hat{A}}$  to the class of the derivation induced by  $\mu$  on  $A$ .*

*Proof.* We take as a projective resolution the bar resolution  $P$  of  $\hat{A}$  where the tensor products are over  $k[[t]]$ :

$$\dots \longrightarrow \hat{A}^{\otimes n+2} \xrightarrow{\delta_n} \hat{A}^{\otimes n+1} \longrightarrow \dots$$

which is given by  $\delta_n(a_0 \otimes \dots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}$ . The last non-zero differential is the map  $\delta_1 : \hat{A}^{\otimes 3} \rightarrow \hat{A}^{\otimes 2}$  which sends  $a \otimes b \otimes c$  to  $ab \otimes c - a \otimes bc$  for  $a, b, c \in \hat{A}$ . We have the following identifications:

$$\begin{aligned} \text{H}^0(\text{Hom}_{\hat{A}^e}(P, \hat{A}/t^r \hat{A})) &= \text{HH}^0(\hat{A}, \hat{A}/t^r \hat{A}) \\ &\cong \text{HH}^0(\hat{A}/t^r \hat{A}) = \text{End}_{\hat{A}^e}(\hat{A}/t^r \hat{A}) \end{aligned} \quad (15)$$

The identity map in  $\text{End}_{\hat{A}^e}(\hat{A}/t^r \hat{A})$  corresponds to the homomorphism

$$\begin{aligned} \zeta : \hat{A} \otimes_{k[[t]]} \hat{A} &\rightarrow \hat{A}/t^r \hat{A} \\ a \otimes b &\mapsto \zeta(a \otimes b) = ab + t^r \hat{A} \end{aligned} \quad (16)$$

for all  $a, b \in A[[t]]$ . This lifts to an  $\hat{A}^e$ -homomorphism

$$\begin{aligned} \bar{\zeta} : \hat{A} \otimes_{k[[t]]} \hat{A} &\rightarrow (\hat{A}/t^{r+1}\hat{A})_\alpha \\ a \otimes b &\mapsto \bar{\zeta}(a \otimes b) = a\alpha(b) + t^{r+1}\hat{A} \end{aligned} \quad (17)$$



for  $a, b \in \hat{A}$  since  $\alpha$  induces the identity on  $\hat{A}/t^r\hat{A}$ .

Since  $\bar{\zeta} \in \text{Hom}_{\hat{A}}(\hat{A} \otimes \hat{A}, (\hat{A}/t^{r+1}\hat{A})_\alpha)$  we need to apply the first non-zero differential

$$\epsilon : \text{Hom}_{\hat{A}^e}(\hat{A}^{\otimes 2}, (\hat{A}/t^{r+1}\hat{A})_\alpha) \rightarrow \text{Hom}_{\hat{A}^e}(\hat{A}^{\otimes 3}, (\hat{A}/t^{r+1}\hat{A})_\alpha) \quad (18)$$

which is given by composing with  $-\delta_1$ . Hence in  $\hat{A}/t^{r+1}\hat{A}$  we have:

$$\begin{aligned} (-\bar{\zeta} \circ \delta_1)(a \otimes b \otimes c) &= -\bar{\zeta}(ab \otimes c + a \otimes bc) = -ab\alpha(c) + a\alpha(bc) \\ &= a(\alpha(b) - b)\alpha(c) = t^r a\mu(b)\alpha(c). \end{aligned} \quad (19)$$

for all  $a, b, c \in \hat{A}$ . We observe that  $t^r a\mu(b)\alpha(c) + t^{r+1}\hat{A} \in \hat{A}/t^{r+1}\hat{A}$  is the image, under  $t^r : \hat{A}/t\hat{A} \rightarrow (\hat{A}/t^{r+1}\hat{A})_\alpha$ , of the map  $\psi : \hat{A}^{\otimes 3} \rightarrow \hat{A}/t\hat{A}$ , that is we have the following commutative diagram:

$$\begin{array}{ccc} & \hat{A}^{\otimes 3} & \\ \psi \swarrow & \downarrow -\zeta \circ \delta & \\ \hat{A}/t\hat{A} & \xrightarrow{t^r} & (\hat{A}/t^{r+1}\hat{A})_\alpha \end{array}$$

where  $\psi$  sends  $a \otimes b \otimes c$  to  $a\mu(b)\alpha(c) + t\hat{A}$  which is equal to  $a\mu(b)c + t\hat{A}$  since  $\alpha(c) - c \in t^r\hat{A} \subseteq t\hat{A}$ . Consequently  $\psi$  induces a map  $\bar{\psi} : \hat{A}^{\otimes 3} \rightarrow \hat{A}$  which sends  $\bar{a} \otimes \bar{b} \otimes \bar{c}$  to  $\bar{a}\bar{\mu}(\bar{b})\bar{c}$  that can be restricted to the map  $\bar{\psi} : \hat{A} \rightarrow \hat{A}$  that sends  $\bar{b}$  to  $\mu(\bar{b})$ . Using (1) the result follows.  $\square$

## 5 Proof of Theorem 1.1

The proof of Theorem 1.1 requires the following result, which is a variation of [6, 5.1]:

**Theorem 5.1.** *Let  $A, B$  be finite-dimensional selfinjective  $k$ -algebras with separable semisimple quotients. Let  $r$  be a positive integer and let  $M, N$  be an  $A$ - $B$ -bimodule,  $B$ - $A$  bimodule, respectively, inducing a stable equivalence of Morita type between  $A$  and  $B$ . Then for any  $\alpha \in \text{Aut}_r(A[[t]])$  there is  $\beta \in \text{Aut}_r(B[[t]])$  such that  ${}_{\alpha^{-1}M}[[t]] \cong M[[t]]_\beta$  as  $A[[t]]$ - $B[[t]]$ -bimodules. This correspondence induce a group isomorphism  $\text{Out}_r(A[[t]]) \cong \text{Out}_r(B[[t]])$  making the following diagram commutative:*

$$\begin{array}{ccc} \text{Out}_r(A[[t]]) & \xrightarrow{\cong} & \text{Out}_r(B[[t]]) \\ \downarrow & & \downarrow \\ HH_r^1(A) & \xrightarrow{\cong} & HH_r^1(B) \end{array}$$

where the vertical maps are from Proposition 3.8 and the lower horizontal isomorphism is induced by the functor  $N \otimes_A - \otimes_A M$

*Proof.* By the Lemma [6, 4.2] we have that the upper horizontal map is a group isomorphism. Let  $\alpha \in \text{Aut}_r(A[[t]])$ ,  $\beta \in \text{Aut}_r(B[[t]])$  such that  ${}_{\alpha^{-1}}M[[t]] \cong M[[t]]_{\beta}$  as  $A[[t]]$ - $B[[t]]$ -bimodules. We also have that  $\alpha$  is such that  $\alpha(a) = a + t^r \mu(a)$  for all  $a \in A[[t]]$  and  $\beta$  such that  $\beta(b) = b + t^r \nu(b)$  for all  $b \in B[[t]]$  for some  $k[[t]]$ -linear endomorphisms  $\mu, \nu$ . We denote by  $\bar{\mu}$  and  $\bar{\nu}$  the classes in  $\text{HH}_r^1(A)$  and  $\text{HH}_r^1(B)$  respectively determined by the canonical group homomorphism  $\text{Out}_r(A[[t]]) \rightarrow \text{HH}^1(A)$  and  $\text{Out}_r(B[[t]]) \rightarrow \text{HH}^1(B)$ . Set  $\hat{M} = M[[t]]$ . By the assumptions, tensoring by  $M$  yields a stable equivalence of Morita type between  $A$  and  $B$ . In particular we have:

$$\text{HH}^1(A) \cong \text{Ext}_{A \otimes_k B^{op}}^1(M, M) \cong \text{HH}^1(B) \quad (20)$$

induced by the functors  $-\otimes_A M$  and  $M \otimes_B -$ . In addition since  $B[[t]]$  is isomorphic to  $\hat{N} \otimes_{A[[t]]} \hat{M}$  in the relatively  $k[[t]]$ -stable category of  $B[[t]] \otimes_{k[[t]]} B[[t]]^{op}$ -modules, it follows that the isomorphism

$$\text{HH}^1(A) \cong \text{HH}^1(B) \quad (21)$$

given by the composition of the two previous isomorphisms is induced by the functor  $N \otimes_A - \otimes_A M$ . The functors  $M \otimes_B -$ ,  $-\otimes_A M$  also induce algebra homomorphisms

$$\text{End}_{A^e}(A) \rightarrow \text{End}_{A \otimes_k B^{op}}(M) \leftarrow \text{End}_{B^e}(B) \quad (22)$$

where  $A^e = A \otimes_k A^{op}$  and similarly for  $B^e$ . Tensoring the following two exact sequence

$$0 \longrightarrow A \longrightarrow (A[[t]]/t^{r+1}A[[t]])_{\alpha} \longrightarrow A[[t]]/t^r A[[t]] \longrightarrow 0$$

and

$$0 \longrightarrow B \longrightarrow (B[[t]]/t^{r+1}B[[t]])_{\alpha} \longrightarrow B[[t]]/t^r B[[t]] \longrightarrow 0$$

by  $-\otimes_{A[[t]]} \hat{M}$  and  $\hat{M} \otimes_{B[[t]]} -$  yields short exact sequences of the form

$$0 \longrightarrow M \rightarrow {}_{\alpha^{-1}}(M[[t]]/t^{r+1}M[[t]]) \rightarrow M \longrightarrow 0$$

$$0 \longrightarrow M \rightarrow (M[[t]]/t^{r+1}M[[t]])_{\beta} \rightarrow M \longrightarrow 0$$

By the naturality properties of the connecting homomorphism and from the description of  $\bar{\mu}$ ,  $\bar{\nu}$  in Proposition 4.1 the image of  $\bar{\mu} \otimes \text{Id}_M$  and  $\text{Id}_M \otimes \bar{\nu}$  in  $\text{Ext}_{A \otimes_k B^{op}}^1(M, M)$  are equal to the images of  $\text{Id}_{\hat{M}}$  under the two connecting homomorphisms

$$\text{End}_{A \otimes_k B^{op}}(\hat{M}) \rightarrow \text{Ext}_{A \otimes_k B^{op}}^1(M, M) \quad (23)$$

obtained after applying the functor  $\text{Hom}_{A[[t]] \otimes B[[t]]^{op}}(\hat{M}, -)$  to the short exact sequences using the same identification used in Proposition 4.1. By the Lemma [6, 4.3] the two exact sequences are equivalent, consequently the connecting homomorphism are equal. Hence the two images of  $\text{Id}_M$  coincide. This shows that the group isomorphism  $\text{HH}_r^1(B) \cong \text{HH}_r^1(A)$  induced by  $\text{Out}_r(B[[t]]) \cong \text{Out}_r(A[[t]])$  is equal to the one determined by the functor  $N \otimes_A - \otimes_A M$ . Hence the result.  $\square$

*Proof of Theorem 1.1.* We show first that the following diagram commutes:

$$\begin{array}{ccc}
\text{Out}_r(A[[t]]) & \xrightarrow{\cong} & \text{Out}_r(B[[t]]) \\
\downarrow (\cdot)^p & & \downarrow (\cdot)^p \\
\text{Out}_{rp}(A[[t]]) & \xrightarrow{\cong} & \text{Out}_{rp}(B[[t]])
\end{array}$$

where the horizontal maps are from Theorem 5.1 and the vertical maps are  $p$ -fold compositions. Let  $\alpha \in \text{Aut}_r(A[[t]])$  and  $\beta \in \text{Aut}_r(B[[t]])$  such that  ${}_{\alpha^{-1}}M[[t]] \cong M[[t]]_{\beta}$ . Let  $\mu, \nu$  be the unique linear maps on  $A[[t]]$  such that  $\alpha(a) = a + t^r \mu(a)$  and  $\beta(b) = b + t^r \nu(b)$  respectively. By Corollary 3.10 we have  $\alpha^p \in \text{Aut}_{rp}(A[[t]])$ ,  $\beta^p \in \text{Aut}_{rp}(B[[t]])$  and also that the maps  $\bar{\mu}, \bar{\nu}$ , induced by  $\mu, \nu$  on  $A$ , is sent under the  $p$ -power map to  $\bar{\mu}^p$  and  $\bar{\nu}^p$  respectively. Hence we have the commutativity of the diagram above since  ${}_{\alpha^{-p}}M[[t]] \cong M[[t]]_{\beta^p}$ . Using the commutative diagram above and Theorem 5.1 we have that the class of  $\bar{\mu}^p$  is sent though the isomorphism defined in Theorem 1.1 to the class of  $\bar{\nu}^p$ . Hence we have the commutativity of the diagram of the Theorem 1.1.  $\square$

## 6 Example

The purpose of the following example is to show that  $p$ -power maps do not commute in general with transfer maps in the Hochschild cohomology of symmetric algebras.

Let  $H = \{1, (123), (132)\} \cong C_3 \leq S_3$  and  $M = kS_3$  considered as a  $kS_3$ - $kC_3$  bimodule. By  $\langle -, - \rangle$  we mean the standard bilinear form for the group algebra  $kH$ . We choose  $\{1, t = (12)\}$  as set of representatives of  $S_3/H$ . We note that  $M$  is finitely generated and projective as a right  $kC_3$ -module, since  $[G : H] = 2$ , so there exist  $x_i \in \text{Hom}_{kC_3}(kS_3, kC_3)$  with  $1 \leq i \leq 2$  such that for any  $x \in M$ ,  $x = \sum_i x_i \varphi_i(x)$ . Explicitly:

$$\begin{aligned}
\varphi_1(1) &= 1, \varphi_1((123)) = (123), \\
\varphi_1((132)) &= (132), \varphi_1(g) = 0
\end{aligned} \tag{24}$$

for every other  $g \in G$ . Similarly we define:

$$\varphi_t(12) = 1, \varphi_t((13)) = (132), \varphi_t((23)) = (123), \varphi_t(g) = 0 \tag{25}$$

for every other  $g \in G$ . Since  $C_3$  is commutative then  $\text{HH}^1(kC_3) = \text{Der}_k(kC_3)$  which is generated by  $\{f_0, f_1, f_2\}$  such that  $f_0((123)) = 1$ ,  $f_1((123)) = (123)$  and  $f_2((123)) = (132)$ . In this case the explicit formula of the transfer map by [5, 2.5] is given by:

$$\begin{aligned}
\text{tr}^M(f) &= \sum_{h \in H} \langle h^{-1}, f(\varphi_1(a)) \rangle h + \langle h^{-1}t, f(\varphi_t(a)) \rangle th + \\
&\quad \langle h^{-1}, f(\varphi_1(at)) \rangle ht + \langle h^{-1}, f(\varphi_t(at)) \rangle tht
\end{aligned} \tag{26}$$

where  $f \in \text{Der}_k(kC_3)$ . In particular for  $a = (123)$  we have:

$$\begin{aligned} \text{tr}^M(f_0)((123)) &= \sum_{h \in H} \left( \langle h^{-1}, f_0((123)) \rangle + \langle h, f_0((132)) \rangle \right) h \\ &= \sum_{h \in H} \left( \langle h^{-1}, 1 \rangle + \langle h, -(123) \rangle \right) h = 1 - (132) \end{aligned} \quad (27)$$

similarly we have:

$$\text{tr}^M(f_0)(132) = 1 - (123). \quad (28)$$

We can note now that  $\text{tr}^M(f_0^{[3]}) = 0$  since  $f_0^{[3]} = 0$ , so  $\text{tr}^M(f_0^{[3]})(132) = 0$ . On the other hand  $\text{tr}^M(f_0)^{[3]}((132)) = \text{tr}^M(f_0) \circ \text{tr}^M(f_0)(1 - (123)) = \text{tr}^M(f_0)(-1 + (132)) = 1 - (123)$ . Since the transfer maps send elements on  $\text{HH}^1(B)$  to elements  $\text{HH}^1(A)$  it should exist an inner derivation in  $S_3$  which sends  $(132)$  to  $1 - (123)$  if we require the commutativity of the diagram. But there is no element in  $a \in kS_3$  such that  $[a, (132)] = 1$ . Hence in this case the  $p$ -power map does not commute with the transfer map.

**Remark 6.1.** This shows that the  $p$ -power map cannot be expressed in terms of the  $BV$ -operator, as this is invariant under transfer maps, by [5, 10.7].

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